4 Analysis of the Orbit

When an orbit has been calculated there will be information about position and velocity where this may be wanted, but there also will be provided the position and velocity defining each reference orbit; all of this information would be referred to some fundamental reference system Suppose that some analysis concerns the times t_a,t_b; then the matrizant $\mathbf{\Omega}(t_a,t_b)$ will be needed If these times lie in the same reference arc, then (2) can be used immediately to give the matrizant If not, suppose that t_i starts the arc containing t_a, t_{i+k} starts the arc containing t_b, and t_{i+1}, , t_{i+k-1} start intermediate arcs Then

$$\mathbf{\Omega}(t_b,t_a) = \mathbf{\Omega}(t_b,t_{i+k})\mathbf{\Omega}(t_{i+k},t_{i+k-1}) \qquad \mathbf{\Omega}(t_{k+2},t_{k+1})\mathbf{\Omega}(t_{k+1},t_a) \quad (6)$$

Each matrizant is calculated using the parameters corresponding to the time that begins the appropriate reference orbit Equation (6) holds even if different arcs are referred to different origins, provided that the orientation of the axes remains the same in space

It also is possible fairly simply to investigate the effects of forces that may have been neglected If these are substituted for **f** in (3), and the parameters for the matrix components used in the evaluation of the integral are those for the reference orbits already found, then the evaluation of (3) will provide a history of the effects that these forces would have had if they had been included in the calculation of the orbit (always assuming these effects to be small) The same applies to small changes in constants used in the work, such as the mass of the moon A small change in a constant will lead to a perturbing force slightly different from the one used in the calculation of the orbit; if this difference is substituted for **f** in (3) the effects of the change can be calculated It may also be possible to investigate unknown forces causing deviation from the calculated motion, if that deviation is known well enough, for then (3) would be treated as an integral equation for the unknown **f**

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The Matrizant of Keplerian Motion

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This paper contains a documentation of various forms of the matrizant of the different types of Keplerian motion (The *matrizant* is the matrix that relates residuals, or small departures from some known orbit, at different times) The formulas can be used in the analysis of Keplerian or nearly Keplerian orbits when relations between coordinates at different times are required

1 Introduction and Notation

TRADITIONALLY the methods of differential correction in celestial mechanics have been used to relate residuals between observed and computed quantities to small changes in the geometrical elements of reference Keplerian orbits These methods cannot be immediately applied when the relations between residuals in position and velocity at two different times are required, and since there is some contemporary preoccupation with these quantities (the geometrical elements being incidental), the author feels that the following forms of the matrizant may prove to be useful Expressions for the matrizants of particular kinds of orbits have been published, but so far as the author is aware, no general survey has appeared in the literature

Keplerian motion is the motion of a particle subject to the force function μ/r Let us assume that some reference Keplerian orbit is completely defined, so that position $\mathbf{r}$ and velocity $\mathbf{r}'$ are known for any time In a "neighboring" Keplerian orbit, position and velocity will be $\mathbf{r} + \delta\mathbf{r}$ and $\mathbf{r}' + \delta\mathbf{r}'$, and the relation between these residuals at times t_0 and t will be given (to the first order in these quantities) by a formula of the kind

$$\begin{bmatrix}\delta\mathbf{r}\\ \delta\mathbf{r}'\end{bmatrix} = \mathbf{\Omega}(t,t_0)\begin{bmatrix}\delta\mathbf{r}_0\\ \delta\mathbf{r}_0'\end{bmatrix} \quad (1)$$

It is assumed that $\delta\mathbf{r}$ is the column matrix

$$\begin{bmatrix}\delta x\\ \delta y\\ \delta z\end{bmatrix}, \text{etc}$$

There are currently many names for the matrix $\mathbf{\Omega}(t,t_0)$ The author prefers "matrizant," as having historical precedence through the work of Peano and Baker (see, for instance, Ref 1) It is convenient to subdivide $\mathbf{\Omega}(t,t_0)$ into four three-by-three matrices, for example,

$$\mathbf{\Omega}(t,t_0) = \begin{bmatrix}\mathbf{L}(t,t_0) & \mathbf{M}(t,t_0)\\ \mathbf{P}(t,t_0) & \mathbf{Q}(t,t_0)\end{bmatrix} \quad (2)$$

In 1932 Bower[2] published a method of differential corrections that, with a few modifications, can be used to compute the matrizant The method applies to elliptic orbits but can be easily broadened to cover hyperbolic orbits also The orientation of the axes is arbitrary However, the components of the matrizant are not given directly but are found from a series of intermediate functions The method will be described below in relation to nearly circular orbits, but most of this paper will deal with explicit formulas

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Myachin[3] has given explicit formulas for the components of $\mathbf{M}(t,t_0)$ [in his notation this is $\mathbf{V}(t,t_0)$], where the coordinate axes are chosen so that the x axis points toward the pericentron, and the y axis toward that point in the orbit for which the true anomaly is 90° This coordinate system will be called the "orbital reference system" in this paper To find the matrix for any other reference system (equatorial, say) it is sufficient to calculate the rotation matrix

$$\mathcal{R} \equiv \begin{bmatrix} P_x & Q_x & R_x \\ P_y & Q_y & R_y \\ P & Q_z & R_z \end{bmatrix} \tag{3}$$

which transforms the coordinates from one system to the other (see, for instance, Ref 4, pp 161, 329) Then the new matrix is given by

$$\mathbf{M}_1 = \mathcal{R}^T \mathbf{M} \mathcal{R} \tag{4}$$

where $\mathcal{R}^T$ is the transpose of $\mathcal{R}$ The formula applies also to the other submatrices If $\mathbf{M}(t,t_0)$ is known analytically, then (see Ref 5, p 289) the other matrices can be found from

$$\begin{aligned} \mathbf{L}(t,t_0) &= -\partial \mathbf{M}(t,t_0)/\partial t_0 \\ \mathbf{P}(t,t_0) &= -\partial^2 \mathbf{M}(t,t_0)/\partial t \partial t_0 \\ \mathbf{Q}(t,t_0) &= \partial \mathbf{M}(t,t_0)/\partial t \end{aligned} \tag{5}$$

The expressions for general t_0,t are rather long, but their calculation can be simplified by the use of the following properties, which follow from the canonical nature of the original equations:

$$\begin{aligned} \mathbf{\Omega}^{-1}(t,t_0) &= \begin{bmatrix} \mathbf{L}(t,t_0) & \mathbf{M}(t,t_0) \\ \mathbf{P}(t,t_0) & \mathbf{Q}(t,t_0) \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \mathbf{Q}^T(t,t_0) & -\mathbf{M}^T(t,t_0) \\ -\mathbf{P}^T(t,t_0) & \mathbf{L}^T(t,t_0) \end{bmatrix} \end{aligned} \tag{6}$$

and

$$\begin{aligned} \mathbf{\Omega}(t,t_0) &= \mathbf{\Omega}(t,T)\,\mathbf{\Omega}(T,t_0) \\ &= \mathbf{\Omega}(t,T)\,\mathbf{\Omega}^{-1}(t_0,T) \\ &= \begin{bmatrix} \mathbf{L}(t,T) & \mathbf{M}(t,T) \\ \mathbf{P}(t,T) & \mathbf{Q}(t,T) \end{bmatrix} \begin{bmatrix} \mathbf{Q}^T(t_0,T) & -\mathbf{M}^T(t_0,T) \\ -\mathbf{P}^T(t_0,T) & \mathbf{L}^T(t_0,T) \end{bmatrix} \end{aligned} \tag{7}$$

Here T is an arbitrary time The first part of (7) follows from the fact that the matrizant is a Jacobian matrix If T is chosen to be a time of pericentron passage, the formulas for the components of the matrizant are considerably simplified The expressions that follow all refer to $\mathbf{L}(t,T)$, etc , and apply to the orbital reference system Results for any t_0,t and for any Cartesian coordinate system can then be found by the use of (7) and (4)

No general expression exists that is valid for every possible orbit, unless $(t - t_0)$ is small enough to permit a power series expansion This possibility is dealt with in Ref 5 If a power series is to be used, then the perturbing forces can easily be included Power series offer particular advantages in the technique of different corrections, when residuals in observations are "filtered" so as to estimate orbital parameters, and the estimate is revised as each set of observations is processed

In Keplerian motion some variable other than the time must appear in the formulas for them to be closed In the case of elliptic motion the variable can be conveniently either the eccentric anomaly E, or the true anomaly f The former is often more practical to use; an advantage of the latter is that the formulas also apply to hyperbolic orbits For hyperbolic orbits it also is possible to use the hyperbolic eccentric anomaly F as a variable Each set of formulas can run into trouble when the orbit is nearly parabolic Also, the orbital reference system becomes poorly defined for a nearly circular orbit All these possibilities are discussed and tabulated below

2 Use of the Eccentric Anomaly

Let

$$\mathbf{L} = \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, \text{etc}$$

Then with respect to the orbital reference system, any quantity with only one 3 as a subscript is zero Let a be the semimajor axis of the reference orbit; then the mean motion n is found from $n^2a^3 = \mu$ Let $S = \sin E$ and $C = \cos E$; then if e is the eccentricity of the reference orbit, the radius vector is given by $r = a(1 - eC)$ The results are given in Table 1

Formulas for hyperbolic orbits can be written down from those for elliptic orbits if the following changes are made: $E \rightrightarrows iF$(so $S \rightrightarrows i\sinh F$, $C \rightrightarrows \cosh F$), $n \rightrightarrows -in_1 = -i(-\mu/a^3)^{1/2}$ (taking a to be negative for a hyperbola), $(1 - e^2)^{1/2} \rightrightarrows i(e^2 - 1)^{1/2}$, where $i^2 = -1$

Normally the independent variable is the time In this case the corresponding value of E or F is found from Kepler's equation or its modification for hyperbolic orbits

3 Nearly Circular Orbits

If an orbit is nearly circular then the rotation matrix $\mathcal{R}$ is poorly determined, as is any value of the eccentric or true anomaly The resulting difficulties are more apparent than real, since, in the form in which the solutions ultimately appear, the uncertain quantities are always multiplied by the eccentricity in one way or another In other words, because the orbit is nearly circular, it matters less where the pericentron direction is anyway (We notice that no trouble can arise if the inclination is small, since neither the argument of pericentron nor the longitude of the node is used)

Assuming the initial data to be position and velocity at some time t_0, the calculation of $\mathcal{R}$ and E_0, the eccentric anomaly at t_0, involves division by e This might be avoided for very small eccentricity if, in the formulas for the matrizant, e is put equal to zero, and the x axis, instead of pointing to the pericentron, is made to point along the radius vector at t_0 Provided that e is of the same order of magnitude as the residuals that are to be considered, the resulting error will be unimportant The calculation of position and velocity in the reference orbit should, of course, remain rigorous with full account taken of the eccentricity

Alternatively, no attempt need be made to choose special axes, and Bower's method can be used Given position $\mathbf{r}_0$ and velocity $\mathbf{r}_0'$ at t_0 the quantities $e\cos E_0$ and $e\sin E_0$ can be found accurately from

$$e\cos E_0 = r_0 \mathbf{r}_0'^2/\mu - 1 \qquad e\sin E_0 = \mathbf{r}_0\,\mathbf{r}_0'/(\mu a)^{1/2}$$

If E corresponds to time t, then $\Delta E = E - E_0$ can be found from $n(t - t_0) = \Delta E + e\sin E_0(1 - \cos\Delta E) - e\cos E_0 \sin\Delta E$, after which $e\cos E$ and $e\sin E$ can be found Then (keeping as far as possible to Bower's notation) the formulas shown in Table 2 can be used to calculate $\mathbf{\Omega}(t,t_0)$ Coordinates of position and velocity in the reference orbit can be found from the vector equations

$$\mathbf{r} = f\mathbf{r}_0 + g\mathbf{r}_0' \qquad \mathbf{r}' = f'\mathbf{r}_0 + g'\mathbf{r}_0'$$

4 Use of the True Anomaly

The formulas in Table 3 can be derived from those in Sec 2 Once found, they apply both to elliptic and to hyperbolic orbits The calculation of f, given the time, must, however, proceed by different paths, depending on the eccentricity To make the formulas valid for all e without any changes, it is convenient to introduce the pericentron distance q and the quantity $\nu = (\mu/q^3)^{1/2}$ Then if $s = \sin f$, $c = \cos f$, $r = q(1 + e)/(1 + ec)$

5 Parabolic and Nearly Parabolic Orbits

If e is nearly equal to one, some components of the matrizant can become large even for moderate $(t - t_0)$ because of the presence of $(1 - e)$ in the denominators This is not due to any fault in the expressions; the formulas are based on a first-order analysis, and first-order theory has its limitations The fact that there is no analytical singularity if $e = 1$ can be demonstrated by giving the components of the matrizant of parabolic motion Letting $w = \tan(f/2)$, the results in Table 4 follow

It is also possible to avoid the appearance of $(1 - e)$ in the denominators by introducing the following quantities:

$$\lambda = (1 - e)/(1 + e)$$

$$\sigma = -6(w^2/3 + w^4/5) + 9\lambda(w^4/5 + w^6/7) - 12\lambda^2(w^6/7 + w^8/9) +$$

Kepler's equation becomes

$$\tfrac{1}{2}(\mu/q^3)^{1/2}(e + 1)^{1/2}(t - T) = w + \tfrac{1}{3}w^3 + \tfrac{1}{3}\lambda\sigma w$$

A typical component of the matrizant is

$$l_{21} = \frac{r}{q(1 + e)}\frac{2w}{(1 + w^2)^2}[\lambda w^4(3 + w^2) - (1 - w^4\lambda^2)\sigma]$$

The series for σ fails to converge in an elliptic orbit on the side of the minor axis containing the apocentron, and here another development could be used But the determination of the eccentric anomaly, given the time, presents no problems in this region, even for high eccentricity, and the use of the formulas of Sec 2 seems to be indicated The components will become large anyway, but this is symptomatic of the type of orbit, and also follows from the use of the time of a pericentron passage as one of the times in the matrizant

For e nearly equal to one it would probably be best, in general, to use the eccentric anomaly and perhaps to increase the precision of the calculations (The true anomaly is impracti-

Table 1 Elliptic orbits: eccentric anomaly

$$l_{11} = \frac{a}{r(1 - e)^2}[C^2(1 + e - e^2) + C(2 + e + 2e^2 - e^3) - 2 - 5e + 2e^2 + 3ES]$$

$$l_{12} = \frac{a}{r(1 - e)}(1 - e^2)^{1/2}S(1 - C)$$

$$l_{21} = \frac{a}{r(1 - e)^2}(1 - e^2)^{1/2}[SC(1 + e) + S(2 - e) - 3EC]$$

$$l_{22} = \frac{a}{r(1 - e)}[C^2 + C(-1 - 2e + e^2) + 1]$$

$$l_{33} = \frac{1}{1 - e}(C - e)$$

$$m_{11} = \frac{a}{nr}(1 - e)S[-C(1 + e) + 2]$$

$$m_{12} = \frac{a}{nr}\frac{(1 - e^2)^{1/2}}{1 - e}[C^2(2 - e) + 2C(1 + e) - 4 - e + 3ES]$$

$$m_{21} = \frac{a}{nr}(1 - e^2)^{1/2}(1 - C)^2$$

$$m_{22} = \frac{a}{nr}[SC(2 + e + e^2) + 2S - 3(1 + e)EC]$$

$$m_{33} = \frac{1}{n}S(1 - e)$$

$$p_{11} = n\left(\frac{a}{r}\right)^3\frac{1}{(1 - e)^2}[SC^2(e + e^2 - e^3) + SC(-2 - 5e + 2e^2) + S(1 + e + 3e^2 - e^3) + 3E(C - e)]$$

$$p_{12} = n\left(\frac{a}{r}\right)^3\frac{(1 - e^2)^{1/2}}{(1 - e)}[eC^3 - 2C^2 + C + 1 - e]$$

$$p_{21} = n\left(\frac{a}{r}\right)^3\frac{(1 - e^2)^{1/2}}{(1 - e)^2}[-C^3(e + e^2) + C^2(2 + 5e) - C(1 + e) - 1 - 3e + e^2 + 3ES]$$

$$p_{22} = n\left(\frac{a}{r}\right)^3\frac{1}{(1 - e)}S[eC^2 - 2C + 1 + e - e^2]$$

$$p_{33} = -n\left(\frac{a}{r}\right)\frac{1}{(1 - e)}S$$

$$q_{11} = \left(\frac{a}{r}\right)^3(1 - e)[C^3(e + e^2) - 2C^2(1 + e) + 2C + 1 - e]$$

$$q_{12} = \left(\frac{a}{r}\right)^3\frac{(1 - e^2)^{1/2}}{1 - e}[SC^2(2e - e^2) - SC(4 + e) + S(1 + 2e + e^2) + 3E(C - e)]$$

$$q_{21} = \left(\frac{a}{r}\right)^3(1 - e^2)^{1/2}S[eC^2 - 2C + 2 - e]$$

$$q_{22} = \left(\frac{a}{r}\right)^3[-C^3(2e + e^2 + e^3) + C^2(4 + 5e + 5e^2) - C(1 + 3e) - 2 - 3e - e^2 + 3(1 + e)ES]$$

$$q_{33} = \left(\frac{a}{r}\right)(1 - e)C$$

Table 2 Nearly circular orbits

$$\mu = k^2$$
$$\tau = k(t - t_0)$$
$$r = a(1 - e\cos E)$$
$$r_0 = a(1 - e\cos E_0)$$
$$F = a(1 - \cos\Delta E)$$
$$f = 1 - F/r_0$$
$$G = a^{1/2}\sin\Delta E$$
$$H = a^{1/2}e\sin E$$
$$J = a^{3/2}\Delta E - aG$$
$$g = \tau - J$$
$$L = (1/r)[3J + 2FH + Gr_0]$$
$$(2M) = (a/r_0)[GL - 2F]$$
$$(2N) = a(-3J + FL)$$
$$(3) = FG/rr_0$$
$$(2) = (2N)/r_0^3 + (3)$$
$$(1) = (1/r_0^3)[G^2r_0/r + (2M) + F]$$
$$(4) = F^2/r$$
$$r' = (k/r)H$$
$$F' = (k/r)G$$
$$f' = 1 - F'/r_0$$
$$G' = (k/r)\cos\Delta E$$
$$H' = (k/r)e\cos E$$
$$J' = a(k/r) - aG$$
$$g' = k - J'$$
$$L' = -r'L/r + (1/r)[3J' + 2F'H + 2FH' + G'r_0]$$
$$(2M)' = (a/r_0)[GL' + G'L - 2F']$$
$$(2N)' = a(-3J' + F'L + FL')$$
$$(3)' = -r'(3)/r + (FG' + F'G)/rr_0$$
$$(2)' = (2N)'/r_0^3 + (3)'$$
$$(1)' = (1/r_0^3)[-G^2r_0r'/r^2 + 2GG'r_0/r + (2M)' + F']$$
$$(4)' = -F^2r'/r^2 + 2FF'/r$$
$$\Phi_0 = \begin{bmatrix} x_0 & x_0' \\ y_0 & y_0' \\ z_0 & z_0' \end{bmatrix}$$
$$\Phi_0^T = \text{the transpose of } \Phi_0$$
$$\mathbf{I} = \text{the identity matrix}$$
$$\mathbf{L} = f\mathbf{I} + \Phi_0\begin{bmatrix} (1) & (3) \\ (2) & (4) \end{bmatrix}\Phi_0^T$$
$$\mathbf{M} = g\mathbf{I} + \Phi_0\begin{bmatrix} (3) & (2M) \\ (4) & (2N) \end{bmatrix}\Phi_0^T$$
$$\mathbf{P} = f'\mathbf{I} + \Phi_0\begin{bmatrix} (1)' & (3)' \\ (2)' & (4)' \end{bmatrix}\Phi_0^T$$
$$\mathbf{Q} = g'\mathbf{I} + \Phi_0\begin{bmatrix} (3)' & (2M)' \\ (4)' & (2N)' \end{bmatrix}\Phi_0^T$$

Table 3 Formulas using the time anomaly

$$l_{11} = \frac{r}{q(1-e)}\,[c^2 + c(2-e) - 2] + \frac{3\nu}{(1-e)(1+e)^{1/2}}\,s(t-T)$$

$$l_{12} = \frac{r}{q(1+e)}\,s(1-c)$$

$$l_{21} = \frac{r}{q(1-e)}\,s(c+2) - \frac{3\nu}{(1-e)(1+e)^{1/2}}\,(c+e)(t-T)$$

$$l_{22} = \frac{r}{q(1+e)}\,[c^2 - c(1-e) + 1]$$

$$l_{33} = \frac{r}{q}c$$

$$m_{11} = \frac{r}{\nu q(1+e)^{3/2}}\,s(-c+2+e)$$

$$m_{12} = \frac{r}{\nu q(1-e)(e+1)^{1/2}}\,2(c-1)(c+2) + \frac{3}{1-e}\,s(t-T)$$

$$m_{21} = \frac{r}{\nu q(1+e)^{3/2}}\,(1-c)^2$$

$$m_{22} = \frac{r}{\nu q(1-e)(e+1)^{1/2}}\,2s(c+1+e) - \frac{3}{1-e}\,(c+e)(t-T)$$

$$m_{33} = \frac{r}{\nu q(e+1)^{1/2}}\,s$$

$$p_{11} = \frac{\nu}{(1-e)(e+1)^{1/2}}\,s[-ec^2 - 2c + 1 - e] + \frac{3\nu^2 q^2}{r^2(1-e)}\,c(t-T)$$

$$p_{12} = \frac{\nu}{(1+e)^{3/2}}\,[-ec^3 - 2c^2 + c + 1 + e]$$

$$p_{21} = \frac{\nu}{(1-e)(e+1)^{1/2}}\,[ec^3 + 2c^2 - c - 1 - e] + \frac{3\nu^2 q^2}{r^2(1-e)}\,s(t-T)$$

$$p_{22} = \frac{\nu}{(1+e)^{3/2}}\,s[-ec^2 - 2c + 1]$$

$$p_{33} = -\frac{\nu}{(1+e)^{1/2}}\,s$$

$$q_{11} = \frac{1}{(1+e)^2}\,[-ec^3 - 2c^2 + 2c + ec + (1+e)^2]$$

$$q_{12} = \frac{1}{(1-e^2)}\,s[-2ec^2 - 4c + 1 - e] + \frac{3\nu q^2}{r^2(1-e)}\,(1+e)^{1/2}c(t-T)$$

$$q_{21} = \frac{1}{(1+e)^2}\,s(1-c)(ec+2+e)$$

$$q_{22} = \frac{1}{(1-e^2)}\,[2ec^3 + 4c^2 - c(1+e) - 2 - e - e^2] + \frac{3\nu q^2}{r^2(1-e)}\,(1+e)^{1/2}s(t-T)$$

$$q_{33} = \frac{1}{1+e}\,(c+e)$$

Table 4 Parabolic orbits

$$l_{11} = (1 + 3w^2 + w^4 - \tfrac{1}{5}w^6)/(1 + w^2)$$
$$l_{12} = 2w^3/(1 + w^2)$$
$$l_{21} = 2w^3(1 + \tfrac{3}{5}w^2)/(1 + w^2)$$
$$l_{22} = (1 + w^4)/(1 + w^2)$$
$$l_{33} = 1 - w^2$$
$$m_{11} = (2^{1/2}/\nu)w(1 + 2w^2)/(1 + w^2)$$
$$m_{12} = (2^{1/2}/\nu)w^4(1 - \tfrac{1}{5}w^2)/(1 + w^2)$$
$$m_{21} = (2^{1/2}/\nu)w^4/(1 + w^2)$$
$$m_{22} = (2^{1/2}/\nu)w(1 + w^2 + \tfrac{6}{5}w^4)/(1 + w^2)$$
$$m_{33} = (2^{1/2}/\nu)w$$
$$p_{11} = 2^{3/2}\nu w(1 + w^2 + \tfrac{1}{5}w^4 - \tfrac{1}{5}w^6)/(1 + w^2)^3$$
$$p_{12} = 2^{1/2}\nu w^2(3 + w^2)/(1 + w^2)^3$$
$$p_{21} = 2^{1/2}\nu w^2(3 + 4w^2 + \tfrac{9}{5}w^4)/(1 + w^2)^3$$
$$p_{22} = 2^{1/2}\nu w(-1 + 2w^2 + w^4)/(1 + w^2)^3$$
$$p_{33} = -2^{1/2}\nu w/(1 + w^2)$$
$$q_{11} = (1 + 5w^2 + 2w^4)/(1 + w^2)^3$$
$$q_{12} = 4w^2(1 + \tfrac{1}{5}w^2 - \tfrac{1}{5}w^4)/(1 + w^2)^3$$
$$q_{21} = 2w^2(2 + w^2)/(1 + w^2)^3$$
$$q_{22} = (1 + 2w^2 + 7w^4 + \tfrac{18}{5}w^6)/(1 + w^2)^3$$
$$q_{33} = 1/(1 + w^2)$$

cable near an asymptote of a hyperbolic orbit, and also near the apocentron of a highly eccentric elliptic orbit) Near the pericentron the determination of the eccentric anomaly, given the time, requires special measures; these are described by Herget (Ref 6, pp 34–37) If e is extremely close to one, it is unlikely that regions very far from the pericentron will be of interest, so that the parabolic formulas can be used for a fair approximation

6 Use of the Matrizant

Applications of the matrizant generally occur through the definition given in Eq (1), and there is implicit in this equation the assumption that the squares and products of all small quantities are negligible In fact, the author has found it to be an adequate test of accuracy to confirm that the squares of the first-order terms should be negligible with respect to accuracy of the calculation If the reference orbit is nearly parabolic, the accuracy of (1) depends critically on the parts of the orbit to which the two times refer It is particularly dangerous to try to predict errors occurring near the pericentron as the result of errors in another part of the orbit This is a limitation of the first-order theory that cannot be avoided It should be emphasized that tests for accuracy involving nearly circular orbits have no relevance to orbits of moderate eccentricity

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